Extensional concepts in intensional type theory, revisited

Krzysztof Kapulkin and Yufeng Li

References

Background

Hofmann, Martin. Extensional constructs in intensional type theory. PhD thesis, 1995.

Kapulkin, Krzysztof and Lumsdaine, Peter LeFanu. The homotopy theory of type theories. Advances in Mathematics, 2018.

Isaev, Valery. Morita equivalences between algebraic dependent type theories. arXiv:1804.05045, 2020.

Main result

Kapulkin, Krzysztof and Li, Yufeng. Extensional concepts in intensional type theory, revisited. Theoretical Computer Science, 2025.

Two Notions of Equalities

Definitional	Propositional
$\vdash a_1 = a_2 : A$	$\vdash p : \mathsf{Id}_A(a_1, a_2)$

Dependent type theory with propositional equality gives intensional type theory (ITT).

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Equality reflection rule

Computation

$$\frac{\vdash a_1 : A \qquad \vdash a_2 : A \qquad \vdash p : \mathsf{Id}_A(a_1, a_2)}{\vdash a_1 = a_2 : A}$$

Adding equality reflection gives extensional type theory (ETT).

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Computation	Logic	Topology
$\dfrac{\begin{array}{c}\vdash a_1 : A \qquad \vdash a_2 : A \\ \vdash p : \mathrm{Id}_A(a_1, a_2)\end{array}}{\vdash a_1 = a_2 : A}$	Provably equal $\Downarrow$ Seems reasonable Definitionally equal	Contractible $\Downarrow$ Not true in general Singleton

Adding equality reflection gives extensional type theory (ETT).

Definitional vs. Propositional Equalities

Substitution vs. transport

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$t = t'$	$p : \mathsf{Id}(t, t')$
$B(t) = B(t')$	$B(t) \xrightarrow{p_*} B(t')$

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- Changing terms between types indexed by propositionally equal terms depends on the proof of equality.

Uniqueness of Identity Proofs (UIP)

$$\frac{\vdash p, p' : \mathsf{Id}_A(a_1, a_2)}{\vdash \mathsf{UIP}(p, p') : \mathsf{Id}(p, p')}$$

Uniqueness of identity proofs

Homotopically discrete space

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Theorem (Hofmann 1995)

ETT is conservative over ITT+UIP.

$$\frac{\vdash p, p' : \mathsf{Id}_A(a_1, a_2)}{\vdash \mathsf{UIP}(p, p') : \mathsf{Id}(p, p')} \quad \leftrightsquigarrow \quad \frac{\vdash p : \mathsf{Id}_A(a_1, a_2)}{\vdash a_1 = a_2 : A}$$

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Limitation. Syntactic result did not account for extensions.

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Need to Determine

1. What is a model of a type theory?
2. A suitable notion of equivalence between categories of models?

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$\operatorname{ob}\mathbb{C} = \coprod_{n\in\mathbb{N}} \operatorname{ob}_n \mathbb{C}$	$\operatorname{ob}_{n+1}\mathbb{C} \xrightarrow{\text{ft}} \operatorname{ob}_n \mathbb{C}$

Notation. If $\operatorname{ft} A = \Gamma$ we write $A = \Gamma.A$.

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Substitutions

$$\begin{tikzcd}
\Delta.f^*A \arrow[r, "f.A"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "\pi"] \\
\Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd}$$

Homotopies in Contextual Categories

$$\frac{\vdash A \text{ Type}}{(x_1, x_2 : A) \vdash \mathrm{Id}_A(x_1, x_2) \text{ Type}}$$

Path object

Provable equality

Definition

A homotopy $H\colon f \sim g$ between $f, g\colon \Gamma \to \Delta \in \mathbb{C}$

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$$\begin{tikzcd}
\Gamma \arrow[rr, "{(f,g)}"] \arrow[dr, "H"'] & & \Delta \times \Delta \\
& \Delta.\Delta.\mathrm{Id}_\Delta \arrow[ur, two heads] &
\end{tikzcd}$$

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Homotopy equivalences $w \colon \Gamma \to \Delta$ are those maps admitting left and right homotopy inverses.

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The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

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Weak type lifting

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Definition

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

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Example (Isaev 2020). The type theories ITT+Unit and ITT+Contr are Morita equivalent.

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- ...then there is an associated model $F\mathbb{C}$ of $\mathbb{T}_2$

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- ...then there is an associated model $F\mathbb{C}$ of $\mathbb{T}_2$
- ...such that if we compile back to $UF\mathbb{C}$ as a model of $\mathbb{T}_1$
- ...then the expressible and provable statements in those two models are correspond propositionally within type theory.

Conservativity of ITT+UIP over ETT, Revisited

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd}
\mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, hook, shift left, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, shift left, "|-|"]
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Proof.

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Proof. All models of ETT also are also models of $\mathsf{ITT} + \mathsf{UIP}$, so there is an inclusion $|-| \colon \mathsf{CxlCat}_{\mathsf{ETT}} \hookrightarrow \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$.

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It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class.

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It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class. The cells are "syntactic":

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Proof. All models of ETT also are also models of ITT + UIP, so there is an inclusion $|-| \colon \mathsf{CxlCat}_{\mathsf{ETT}} \hookrightarrow \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$. By cocompleteness, it has a left adjoint $\langle - \rangle$.

It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class. The cells are "syntactic": obtained by freely adding types and terms but no definitional equalities.

Conservativity of ITT+UIP over ETT, Revisited

Theorem

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It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class. The cells are "syntactic": obtained by freely adding types and terms but no definitional equalities. This makes it tractable to explicitly construct $\langle \mathbb{C} \rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$. ♡

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

A quotient construction

- To support equality reflection: must identify homotopic maps.

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  - Example. The map Bool → Bool swapping true and false is a propositional isomorphism but is not the identity even under equality reflection.
- Upshot. $\langle\mathbb{C}\rangle$ is obtained from $\mathbb{C}$ by carefully choosing a wide subcategory of homotopy equivalences to collapse.

Conclusion

Future directions

- Constructive proof of Hofmann's result.
- Encompassing internal universes.
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Thank you!